8.1. Actions of discrete groups. Let Γ be a group acting on a topological space X by homeomorphisms. We also assume X to be Hausdorff (this hypothesis was missing in the exercise sheet, but we use it in the solution).

(a) Show that Γ acts properly discontinuously if and only if the map $\Gamma \times X \to X \times X$ given by $(\gamma, x) \mapsto (\gamma x, x)$ is *proper*, that is, the inverse image of any compact set is compact. Here Γ is equipped with the discrete topology.

(b) Show that if X is a topological manifold and Γ acts freely and properly discontinuously, then the quotient space X/Γ is a topological manifold and the projection $\pi\colon X \to X/\Gamma$ is a covering map. (We assume topological manifolds to be Hausdorff.)

(c) Suppose now that X, M are connected topological manifolds, $F\colon X \to M$ is a covering map, and $\Gamma \subset \mathrm{Homeo}(X)$ is the group of deck transformations of F. Show that Γ acts freely and properly discontinuously on X.

Solution. (a) The inverse image of a compact subset of $X \times X$ of the form $K \times K$, where $K \subset X$ is compact, is the set of all pairs $(\gamma, x) \in \Gamma \times X$ with $\gamma x, x \in K$ or, equivalently, the union of the sets $\{\gamma\} \times \gamma^{-1}(\gamma K \cap K)$ such that $\gamma K \cap K \neq \emptyset$. Since $\gamma^{-1}(\gamma K \cap K)$ is compact, the union is compact if and only if there are only finitely many such γ.

(b) Let $x \in X$, and let $U \subset X$ be the domain of a manifold chart around x with compact closure. Since Γ acts properly discontinuously, there exist only finitely many distinct $\gamma_1, \dots, \gamma_n \in \Gamma$ with $\gamma_i U \cap U \neq \emptyset$. Since the action is free, the points $\gamma_i x$ are distinct, so using that X is Hausdorff we find a smaller open neighborhood $V \subset U$ of x such that $\gamma_1 V, \dots, \gamma_n V$ are pairwise disjoint. The projection π maps each $\gamma_i V$ homeomorphically onto its image in the quotient. Since x was arbitrary, π is a covering map.

It remains to show that X/Γ is Hausdorff. Let $x, y \in X$ with $\pi(x) \neq \pi(y)$. Similarly as above, there exists a compact neighborhood K' of x that is disjoint from $\gamma K'$ for all $\gamma \neq \mathrm{id}$. Since the action is properly discontinuous, the set of all γ with $\gamma(K' \cup \{y\}) \cap K' \neq \emptyset$, or $\gamma y \in K'$, is finite. Hence, there exists a compact neighborhood $K \subset K'$ of x such that $K \cap \Gamma y = \emptyset$. Analogously, there exists a compact neighborhood L of y with $\gamma L \cap L = \emptyset$ for all $\gamma \neq \mathrm{id}$ and $L \cap \Gamma x = \emptyset$. Now $K \setminus \Gamma L$ is a neighborhood of x, $L \setminus \Gamma K$ is a neighborhood of y, and $\pi(K \setminus \Gamma L)$ and $\pi(L \setminus \Gamma K)$ are disjoint neighborhoods of $\pi(x)$ and $\pi(y)$, respectively.

(c) By definition, all $\gamma \in \Gamma$ satisfy $F \circ \gamma = F$; in particular, every orbit Γx is contained in the fiber $F^{-1}\{F(x)\}$.

We show first that for every pair of points $(x_1, x_2) \in X \times X$ there exist neighborhoods U_1 of x_1 and U_2 of x_2 such that $\gamma U_1 \cap U_2 \neq \emptyset$ for at most one $\gamma \in \Gamma$. If $F(x_1) \neq F(x_2)$, this follows from the Hausdorff property of M. Assume now that $F(x_1) = F(x_2) = p$. Let $V \subset M$ be a connected open neighborhood of p such that $F^{-1}(V)$ is a union of disjoint open sets each of which is mapped homeomorphically onto V by F, and let U_1 and U_2 be the respective open sets containing x_1 and x_2. If $\gamma U_1 \cap U_2 \neq \emptyset$, then necessarily $\gamma x_1 = x_2$. There is at most one such γ: if also $\gamma' x_1 = x_2$, and y is any point in X, take a path α from x_1 to y and lift $F \circ \alpha$ to x_2 to show that $\gamma' y = \gamma y$.

Now let K be any compact subset of X. Then $K \times K$ is compact and has a finite covering by product neighborhoods of the form $U_1 \times U_2$ with $\gamma U_1 \cap U_2 \neq \emptyset$ for at most one $\gamma \in \Gamma$. Therefore, the set of all $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$ is finite, and Γ acts freely and properly discontinuously.

□

8.2. Translations. Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

(a) Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

(b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

Solution. (a) For each $g \in \Gamma$ there is some $v_g \in \mathbb{R}^m$ such that $gx = x + v_g$ for all $x \in \mathbb{R}^m$ and since Γ acts freely, we have $v_g \neq 0$ for $g \neq \mathrm{id}$. We denote $V := \{v_g \in \mathbb{R}^m : g \in \Gamma\}$. Note that, as Γ acts properly discontinuously, $V \cap B_r(0)$ is finite for all $r > 0$ and thus each subset of V has an element of minimal length.

We now do induction on m. For $m = 1$, choose $g \in \Gamma \setminus \{\mathrm{id}\}$ such that $|v_g|$ is of minimal length. If there is some $v \in V$ with $v = \lambda v_g$, $\lambda \notin \mathbb{Z}$, we also have $w := v - \lfloor \lambda \rfloor v_g \in V \setminus \{0\}$ with $|w| < |v|$, a contradiction to minimality.

For $m \geq 2$, let $v_g \in V \setminus \{0\}$ be of minimal length and let $V' := \mathrm{span}(v_g) \cap V$. By the same argument as above, we get $V' = \mathbb{Z} v_g$.

Then we have $\mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R}v_g$ with projection map $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ and $\Gamma' := \Gamma/g\mathbb{Z}$ acts by translations on $\mathbb{R}^{m-1}$ via $[h]x = x + \pi(v_h)$. As for $h \notin g\mathbb{Z}$ we have $\pi(v_h) \neq 0$, this action is free. We claim that it is properly discontinuous as well. If not, there are $(h_n)_{n\in\mathbb{N}} \in \Gamma$ with $\pi(v_{h_n}) \neq \pi(v_{h_{n'}})$ and $|\pi(v_{h_n})| < r$ for some $r > 0$. But then, there are $l_n \in \mathbb{Z}$ such that $|v_{h_n} - \pi(v_{h_n}) - l_n v_g| < |v_g|$, i.e. $(v_{h_n - l_n g})_{n\in N}$ is an infinite subset of $V \cap B_{r+|v_g|}(0)$, contradicting that Γ acts properly discontinuously.

By our induction hypothesis, there are $h_2, \ldots, h_k \in \Gamma$ such that

$$\pi(V) = \mathbb{Z}\pi(v_{h_2}) \oplus \ldots \oplus \mathbb{Z}\pi(v_{h_k})$$

and consequently $V = \mathbb{Z}v_g \oplus \mathbb{Z}v_{h_2} \oplus \ldots \oplus \mathbb{Z}v_{h_k}$.

(b) Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^m/\Gamma$ denote the covering map and let $c\colon [0,1] \to \mathbb{R}^m/\Gamma$ be a closed curve in $\mathbb{R}^m/\Gamma$. Then for $p \in \pi^{-1}(c(0))$, there exists a unique lift $\bar{c}\colon [0,1] \to \mathbb{R}^m$ of c with $\bar{c}(0) = p$. Furthermore, if c is not null-homotopic, we have $q := \bar{c}(1) \neq \bar{c}(0)$ and therefore

$$L(c) = L(\bar{c}) \geq d(p,q) = \left|\sum_{i=1}^{k} z_i v_i\right|,$$

for some $(z_1, \ldots, z_k) \in \mathbb{Z}^k \setminus \{0\}$.

Finally, if $v = \sum_{i=1}^{k} z_i v_i \neq 0$ is of minimal length, then $c\colon [0,1] \to \mathbb{R}^m/\Gamma$, $c(t) := \pi(tv)$, has length $L(c) = |v|$.

□

8.3. Some consequences of non-positive sectional curvature. Let M be a Hadamard manifold. Prove the following:

(a) For each $p \in M$, the map $(\exp_p)^{-1}\colon M \to TM_p$ is 1-Lipschitz.

(b) For $p, x, y \in M$, it holds

$$d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma \leq d(x,y)^2,$$

where γ denotes the angle in p.

(c) Let M be a complete manifold of non-positive sectional curvature with universal covering $\pi\colon \tilde{M} \to M$. Assume the dimension of M is greater or equal to 2 (this hypothesis was missing in the exercise sheet, but it is needed: consider $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$). If there is a geodesic $\tilde{\gamma}\colon \mathbb{R} \to \tilde{M}$ which is invariant under deck transformations, then M is not compact.

Solution. (a) By the Theorem of Hadamard-Cartan, we know that $\exp_p$ is a diffeomorphism, hence $(\exp_p)^{-1}$ is well defined.

For $x, y \in M$, let $\bar{x} := (\exp_p)^{-1}(x)$ and $\bar{y} := (\exp_p)^{-1}(y)$. Furthermore, let $c\colon [0, l] \to M$ be the (minimizing) geodesic from x to y. Then by Corollary 3.21, we get

$$d(\bar{x}, \bar{y}) \leq L((\exp_p)^{-1} \circ c) \leq L(c) = d(x, y).$$

(b) We denote again $\bar{x} := (\exp_p)^{-1}(x)$ for $x \in M$. Then this follows directly from (a) and the law of cosines in TM_p:

$$\begin{aligned} d(x, y)^2 &\geq d(\bar{x}, \bar{y})^2 \\ &= d(\bar{p}, \bar{x})^2 + d(\bar{p}, \bar{y})^2 - 2d(\bar{p}, \bar{x})d(\bar{p}, \bar{y}) \cos \bar{\gamma} \\ &= d(p, x)^2 + d(p, y)^2 - 2d(p, x)d(p, y) \cos \gamma. \end{aligned}$$

(c) Let $\tilde{p} = \tilde{\gamma}(0)$and $\tilde{\beta}\colon \mathbb{R} \to \tilde{M}$ a geodesic with $\tilde{\beta}(0) = \tilde{p}$ and $\dot{\tilde{\gamma}}(0) \perp \dot{\tilde{\beta}}(0)$. Furthermore, we denote $\beta := \pi \circ \tilde{\beta}$, $\gamma := \pi \circ \tilde{\gamma}$ and $p := \pi(\tilde{p})$.

For $t > 0$, consider now a minimizing geodesic $\alpha_t\colon [0, l] \to M$ joining $\beta(t)$ to p. We claim that $L(\alpha_t) \geq t$. Indeed, let $\tilde{\alpha}_t$ be a lift of α_t with $\tilde{\alpha}_t(0) = \tilde{\beta}(t)$. As $\tilde{\gamma}$ is invariant under deck transformations, we get $\tilde{\beta}(l) \in \tilde{\gamma}(\mathbb{R})$. From part (b) it follows that

$$L(\alpha_t) = L(\tilde{\alpha}_t) \geq d(\tilde{\alpha}_t(0), \tilde{\alpha}_t(l)) \geq d(\tilde{\beta}(t), \tilde{\beta}(0)) = L(\tilde{\beta}|_{[0,t]}) = t.$$

As $t > 0$ was chosen arbitrarily, we get that M is unbounded.

$\square$